DYNAMICS OF AUTOMORPHISMS OF COMPACT COMPLEX MANIFOLDS

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ABSTRACT. We give an algebro-geometric approach towards the dynamics of automorphisms/endomorphisms of projective varieties or compact Kähler manifolds, try to determine the building blocks of automorphisms /endomorphisms, and show the relation between the dynamics of automorphisms/endomorphisms and the geometry of the underlying manifolds.

1. INTRODUCTION

2. INTRODUCTION

I will report recent results obtained in [KOZ], [NZ], [Zh1], [Zh2], [Zh3], some being jointly with J. Keum, N. Nakayama and K. Oguiso.

Readers may first skip §2 and go directly to subsequent sections. For entropy and dynamical degree, there is a short survey [Fr06]. For (dynamics of polarized) endomorphisms, please see surveys [Zs] and [FN].

We work over the field $\mathbb{C}$ of complex numbers.

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3. Preliminaries: entropy, dynamical degree

We use the convention of Hartshorne's book, [KMM] and [KM].

Let X be a compact complex Kähler manifold with

$$H^*(X, \mathbb{C}) = \bigoplus_{i \geq 0} H^i(X, \mathbb{C})$$

the total cohomology group. Take a $g \in \mathrm{Aut}(X)$. Denote by $\rho(g)$ the *spectral radius* of $g^*|H^*(X, \mathbb{C})$. It is known that either $\rho(g) > 1$, or $\rho(g) = 1$ and all eigenvalues of $g^*|H^*(X, \mathbb{C})$ are of modulus 1. When $\log \rho(g) > 0$ (resp. $\log \rho(g) = 0$) we say that g is of *positive entropy* (resp. *null entropy*).

We refer to Gromov [Gr], Yomdin [Yo], Friedland [Fr95], and Dinh - Sibony [DS05], for the definition of the i-th *dynamical degree* $d_i(g)$ for $1 \leq i \leq n = \dim X$ (note that $d_n(g) = 1$ now and we set $d_0(g) = 1$) and the actual definition of the *topological entropy* $h(g)$ which turns out to be $\log \rho(g)$ in the current setting.

Remark 31. The above terminlogies can also be defined for endomorphisms and even for dominant meromorphic self-maps.

Let Y be a projective variety and $g \in \mathrm{Aut}(Y)$. We say that g is of *positive entropy*, or *null entropy*, or *parabolic*, or *periodic*, or *rigidly parabolic*, or of *primitively positive entropy* (see the definitions below), if so is $g \in \mathrm{Aut}(\tilde{Y})$, where $\tilde{Y} \to Y$ is one (and hence all) g-equivariant resolutions guaranteed by Hironaka. The definitions do not depend on the choice of $\tilde{Y}$ because every two g-equivariant resolutions are birationally dominated by a third one.

We use $g|Y$ to signify that $g \in \mathrm{Aut}(Y)$.

By a *pair* (Y, g) we mean a projective variety Y and an automorphism $g \in \mathrm{Aut}(Y)$. Two pairs (Y', g) and (Y'', g) are *birationally equivariant*, if there is a birational map $\sigma : Y' \cdots \to Y''$ such that the action $\sigma(g|Y')\sigma^{-1} : Y'' \cdots \to Y''$ is biregular.

$g \in \mathrm{Aut}(Y)$ is *periodic* if the order $\mathrm{ord}(g)$ is finite. g is *parabolic* if $\mathrm{ord}(g) = \infty$ and if g is of null entropy.

(Y', g) is *rigidly parabolic* if ($g|Y'$ is parabolic and) for every pair (Y, g) which is birationally equivariant to (Y', g) and for every g- equivariant surjective morphism $Y \to Z$ with $\dim Z > 0$, we have $g|Z$ parabolic.

Let Y' be a projective variety and $g \in \mathrm{Aut}(Y')$ of positive entropy (so $\dim Y' \geq 2$). A pair (Y', g) is of *primitively positive entropy* if it is not of imprimitive positive entropy, while a pair (Y', g) is of *imprimitively positive entropy* if it is birationally equivariant to a pair (Y, g) and if there is a g-equivariant surjective morphism $f : Y \to Z$ such that either one of the two cases below occurs.

(a) $0 < \dim Z < \dim Y$, and $g|Z$ is still of positive entropy.
(b) $0 < \dim Z < \dim Y$, and $g|Z$ is periodic.

Remark 32. We observe that in Case(b), for some $s > 0$ we have $g^s|Z = \mathrm{id}$ and that g^s acts faithfully on the general fibre Y_z of $Y \to Z$, such that

$g^s|Y_z$ is of positive entropy; see [Zh2, (2.1)(11)]. In fact, we have $d_1(g^s|Y) = d_1(g^s|Y_z)$; see [NZ, Appendix, Theorem A.10].

Remark 33. In view of the observation above, if $\dim Y \le 2$ and if the pair (Y, g) is of positive entropy, then $\dim Y = 2$ and the pair (Y, g) is always of primitively positive entropy.

For the references to the following important results, see Dinh-Sibony [DS05] and Guedj [Gu05].

Theorem 34. *Let M be a compact Kähler manifold of dimension n and let g be an automorphism of M. By $\rho(g^*|W)$ we denote the spectral radius of the action of g^* on a g^*-stable subspace W of the total cohomology group $H^*(M, \mathbb{C})$. Then we have*:

(1) *$\rho(g^*|H^*(M, \mathbb{C})) \ge 1$, and $\rho(g^*|H^*(M, \mathbb{C})) = 1$ (resp. > 1) if and only if $\rho(g^*|H^2(M, \mathbb{C})) = 1$ (resp. > 1). Further, $\rho(g^*|H^*(M, \mathbb{C})) = 1$ (resp. > 1) if and only if so is for g^{-1}.*

(2) *One has*

$$\rho(g^*|H^2(M, \mathbb{C})) = \rho(g^*|H^{1,1}(M)).$$

Further, if M is projective, then the above value equals $\rho(g^*|\operatorname{NS}(M))$.

(3) *The spectral radius $\rho(g^*|H^{i,i}(X, \mathbb{C}))$ equals the i-th dynamical degree $d_i(g)$. Further, there are integers $m \le m'$ such that*

$$1 = d_0(g) < \cdots < d_m(g) = \cdots = d_{m'}(g) > \cdots > d_n(g) = 1.$$

(4) *The following holds with $h(g)$ the topological entropy of g*:

$$\rho(g^*|H^*(X, \mathbb{C})) = \max_{0 \le i \le n} d_i(g) = e^{h(g)}.$$

4. Gizatullin-Harbourne-McMullen conjecture

Consider the following question of Gizatullin-Harbourne-McMullen (see [Ha, page 409] and [Mc07, §12]), where $\operatorname{Aut}^*(X) := \operatorname{Im}(\operatorname{Aut}(X) \to \operatorname{Aut}(\operatorname{Pic}(X)))$:

Question 41. Let X be a smooth projective rational surface. If $\operatorname{Aut}^*(X)$ is infinite, is there then a birational morphism φ of X to a surface Y having an anti-pluricanonical curve and an infinite subgroup $G \subset \operatorname{Aut}^*(Y)$ such that G lifts via φ to X?

Remark 42. It is very difficult to construct examples of automorphisms g on rational surfaces X with positive entropy. There are some sporadic examples by A. Coble and M. Gizatullin. In [Mc07], McMullen succeeded in constructing an infinite series of such examples (X_n, g_n). In all the examples of his, X_n has an anti-canonical curve. This lead him to ask a question similar to the one above. In [Zh1], we show that $\operatorname{Aut}(X_n) = (\text{finite group}) \rtimes \mathbb{Z}$, so each of McMullen's surfaces essentially has only one automorphism of positive entropy.

A member in an anti-pluricanonical system $|-nK_X|$ ($n \ge 1$) is called an *anti n-canonical* curve (or *divisor*) or an *anti-pluricanonical* curve; a member in $|-K_X|$ is an *anti* 1*-canonical* curve, or an *anti-canonical* curve.

The result below answers Question 41 in the case of null entropy.

Theorem 43 (cf. [Zh1])**.** *Let X be a smooth projective rational surface and $G \le \mathrm{Aut}(X)$ an infinite subgroup of null entropy. Then we have*:

(1) *There is a G-equivariant smooth blowdown $X \to Y$ such that $K_Y^2 \ge 0$ and hence Y has an anti-pluricanonical curve.*
(2) *Suppose further that* $\mathrm{Im}(G \to \mathrm{Aut}(\mathrm{Pic}(X)))$ *is also an infinite group. Then the Y in* (1) *can be so chosen that $-K_Y$ is nef of self intersection zero and Y has an anti* 1*-canonical curve.*

For groups which are not necessarily of null entropy, we have the following result which is especially applicable (with the same kind of H) when $G/H \ge \mathbb{Z} \rtimes \mathbb{Z}$.

Theorem 44 (cf. [Zh1])**.** *Let X be a smooth projective surface and $G \le \mathrm{Aut}(X)$ a subgroup. Assume that there is a sequence of groups*

$$H \lhd A \lhd G$$

satisfying the following three conditions:

(1) $\mathrm{Im}(H \to \mathrm{Aut}(\mathrm{NS}(X)))$ *is finite;*
(2) *A/H is infinite and abelian; and*
(3) $|G/A| = \infty$.

Then G contains a subgroup S of null entropy and infinite order.
In particular, when X is rational, there is an S-equivariant smooth blowdown $X \to Y$ such that Y has an anti-pluricanonical curve.

Remark 45.

(1) Conditions like the ones in Theorems 43 and 44 are necessary in order to have an affirmative answer to Question 41. See Bedford-Kim [BK, Theorem 3.2] for a pair (X, g) with g of positive entropy and Iitaka D-dimension $\kappa(X, -K_X) = -\infty$.
(2) The blowdown process $X \to Y$ to the minimal pair (Y, S) in Theorems 43 and 44 is necessary, as observed by Harbourne [Ha].

5. Tits alternative for automorphism group

We often study a group G of automorphisms of a projective variety X through its action on cohomological spaces such as on the Néron-Severi group (over $\mathbb{Q}$) $\mathrm{NS}_{\mathbb{Q}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Set

$$G^* = \mathrm{Im}(G \longrightarrow \mathrm{GL}\,(\mathrm{NS}_{\mathbb{Q}}(X))).$$

Then the famous Tits Alternative Theorem says:

(i) *either G^* contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$ of rank two (highly noncommutative); or*
(ii) *G^* contains a connected solvable subgroup G_1 of finite index.*

Here, G_1 is *connected* if its Zariski closure in $\mathrm{GL}\,(\mathrm{NS}_{\mathbb{C}}(X))$ is connected.

In [DS04], Dinh and Sibony proved the following inspiring result:

Theorem 51 ([DS04])**.** *Let M be an n-dimensional compact Kähler manifold. Let G be an abelian subgroup of* $\mathrm{Aut}\,(M)$ *such that each element of $G \setminus \{\mathrm{id}\}$ is of positive entropy. Then G is a free abelian group of rank at most $n - 1$. Furthermore, the rank estimate is optimal.*

In view of the Tits Alternative Theorem and Dinh-Sibony's Theorem, it is natural to pose the following conjecture:

Conjecture 52. (Conjecture of Tits type)
Let X be an n-dimensional compact Kähler manifold or an n-dimensional complex projective variety with at most rational $\mathbb{Q}$-factorial singularities. Let G be a subgroup of $\mathrm{Aut}\,(X)$. Then, one of the following two assertions holds:

(1) *G contains a subgroup isomorphic to $\mathbb{Z} * \mathbb{Z}$.*
(2) *There is a finite-index subgroup G_1 of G such that the subset*

$$N(G_1) = \{g \in G_1 \,|\, g \text{ is of null entropy}\}$$

of G_1 is a normal subgroup of G_1 and the quotient group $G_1/N(G_1)$ is a free abelian group of rank at most $n - 1$.

It turns out that the crucial part of the conjecture is the rank estimate in the statement (2), as a weaker version of the conjecture can be easily verified.

In [KOZ], we prove the result below. In the process, we need to utilize Birkhoff's generalized Perron-Frobenius theorem and show a Lie-Kolchin

type theorem for nef cones; some geometrical analysis of Calabi-Yau threefolds is also necessary.

Theorem 53 (cf. [KOZ]). *The conjecture of Tits type for a group G on a compact complex variety X holds in the following cases.*

(1) $\dim X \le 2$.
(2) *X is a Hyperkähler manifold.*
(3) *X is a complex torus.*
(4) *X is a projective minimal threefold (with at worst $\mathbb{Q}$-factorial terminal singularities and nef canonical divisor).*

6. Dynamics of automorphisms

In this section, we show that the dynamics of automorphisms on all projective complex manifolds (of dimension 3, or of any dimension but assuming the Good Minimal Model Program or Mori's Program) are canonically built up from the dynamics on just three types of projective complex manifolds X (here X is *wCY* if $\kappa(X) = 0 = q(X)$; see [Zh2], [NZ]):
Complex Torus, wCY, Rationally Connected Manifold.

Theorem 61 (cf. [Zh2]). *Let X be a smooth projective complex manifold of $\dim X \ge 2$, and with $g \in \mathrm{Aut}(X)$. Then we have:*

(1) *Suppose that (X, g) is either rigidly parabolic or of primitively positive entropy. Then the Kodaira dimension $\kappa(X) \le 0$.*
(2) *Suppose that $\dim X = 3$ and g is of positive entropy. Then $\kappa(X) \le 0$, unless $d_1(g^{-1}) = d_1(g) = d_2(g) = e^{h(g)}$ and it is a Salem number. Here $d_i(g)$ are dynamical degrees and $h(g)$ is the entropy.*

In the above, an algebraic integer $\lambda > 1$ of degree $2(r+1)$ over $\mathbb{Q}$ with $r \ge 0$, is a *Salem* number if all conjugates of λ over $\mathbb{Q}$ (including λ itself) are given as follows, where $|\alpha_i| = 1$:

$$\lambda,\ \lambda^{-1},\ \alpha_1,\ \bar{\alpha}_1, \ldots, \alpha_r,\ \bar{\alpha}_r.$$

In view of Theorem 61, we have only to treat the dynamics on those X with $\kappa(X) = 0$ or $-\infty$. This is done in [Zh2]. Below is an sample result for threefolds, which says that 3-dimensional dynamics of positive entropy (not necessarily primitive) are just those of 3-tori, wCY 3-folds and rational connected 3-folds, unless dynamical degrees are Salem numbers.

Theorem 62 (cf. [Zh2]). *Let X' be a smooth projective complex threefold. Suppose that $g \in \mathrm{Aut}(X')$ is of positive entropy. Then there is a pair (X, g) birationally equivariant to (X', g), such that one of the cases below occurs.*

(1) *There exist 3-torus $\tilde{X}$ and g-equivariant étale Galois cover $\tilde{X} \to X$.*
(2) *X is a wCY.*
(3) *X is a rationally connected threefold in the sense of Campana and Kollar-Miyaoka-Mori.*
(4) $d_1(g^{\pm}|X) = d_2(g^{\pm}|X) = e^{h(g|X)}$ *and it is a Salem number.*

We refer to McMullen [Mc02a], [Mc02b], [Mc07] for the relation between Salem numbers and K3 surfaces or anti-canonical rational surfaces. See also McMullen [Mc02b] and Cantat [Ca] for the systematic study of dynamics on K3 surfaces.

7. Building blocks of endomorphisms

We will prove in this section the claim below.

Claim 71. All étale endomorphisms of projective manifolds X are constructed from two building blocks below (up to isomorphism), assuming the good minimal model conjecture (known up to dimension three):

Endomorphism of abelian varieties, and
Nearly étale rational endomorphisms of weak Calabi-Yau varieties.

See Definitions 77 and 78 below for the notion above. The étaleness assumption of X is quite natural because every surjective endomorphism of X is étale provided that X is a nonsingular projective variety and is non-uniruled.

We like to use building blocks, via 'canonical fibration', to construct canonically all étale endomorphisms. The canonicity here roughly corresponds to the equivariance of a fibration or morphism. On a nonsingular projective variety X, there are three such canonical fibrations:

The Iitaka fibration for those X of Kodaira dimension $\kappa(X) > 0$,
The albanese map for those X with irregularity $q(X) > 0$, and
The maximal rationally connected fibration for uniruled varieties.

Our reduction of endomorphisms to the building blocks will go along the line of these canonical fibrations. But we need to take care of the equivariance which is not always true.

We start with the result below for those X of positive Kodaira dimension.

Theorem 72 below treats not only holomorphic surjective endomorphisms of projective varieties of $\kappa > 0$ but also dominant meromorphic endomorphisms of compact complex manifolds of $\kappa > 0$ in the class $\mathcal{C}$ in the sense of Fujiki. Note that a compact complex manifold is in the class $\mathcal{C}$ if and only if it is bimeromorphic to a compact Kähler manifold.

Theorem 72 (cf. [NZ, Theorem A])**.** *Let X be a compact complex manifold in the class $\mathcal{C}$ of $\kappa(X) \geq 1$ and let $f\colon X \dashrightarrow X$ be a dominant meromorphic map. Let W_m be the image of the m-th pluricanonical map*

$$\Phi_m\colon X \dashrightarrow |mK_X|^{\vee} = \mathbb{P}(\mathrm{H}^0(X, mK_X))$$

giving rise to the Iitaka fibration of X. Then there is an automorphism g of W_m of finite order such that $\Phi_m \circ f = g \circ \Phi_m$.

Remark 73. If f is holomorphic, then, resolving the indeterminacy points of Φ_m, we may assume that both $f\colon X \to X$ and $\Phi_m\colon X \to W_m$ are holomorphic so that $\Phi_m \circ f = g \circ \Phi_m$. This is because f is étale and we can take an equivariant resolution of the graph of Iitaka fibration.

With Theorem 74 below, we can see the reductive significance of Theorem 72 above from the dynamics point of view. Indeed, in Theorem 72, if we assume that f is holomorphic then f is étale ($\kappa(X)$ being non-negative now); we may assume that $g^s = \mathrm{id}$ (for some $s > 0$) so that $\Phi_m \circ f^s = \Phi_m$; thus the topological entropies and the first dynamical degrees satisfy

$$(h_{\mathrm{top}}(f))^s = h_{\mathrm{top}}(f^s|_F), \qquad (d_1(f))^s = d_1(f^s|_F)$$

for a smooth fibre F of Φ_m (see the remark above).

Theorem 74 (cf. [NZ, Appendix A: Th D, Th A.10])**.**
Let $\pi : X \to Y$ be a fibre space from a compact Kähler manifold X onto a compact complex analytic variety Y and let $f : X \to X$ be an étale surjective endomorphism such that $\pi \circ f = \pi$. Then we have:

(1) *The equality $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(f|_F)$ holds for the topological entropies h_{top} of f and its restriction $f|_F : F \to F$ to a smooth fibre of π.*
(2) *The equality $d_1(f) = d_1(f|_F)$ holds for the first dynamical degrees d_1 of f and $f|_F$ (the étaleness of f is not needed here).*

In view of Theorem 72 above, we are reduced to considering endomorphisms $f : X \to X$ on those nonsingular projective varieties X of Kodaira dimension zero (like the fibres F of the Iitaka fibration in Theorem 72).

Now let X be a nonsingular projective variety of Kodaira dimension zero. One would naturally try to reduce the study of $f : X \to X$ to that of the induced self map on a minmal model $X_{\min}$ of X.

Indeed, the existence of a good minimal model in the sense of Kawamata is known in dimension three or less. In higher dimension, there has been rapid progress due to Birkar, Cascini, Hacon and McKernan [BCHM].

The trouble is, on the one hand, that an holomorphic endomorphism $f : X \to X$ induces only a rational map $f_{X_{\min}} : X_{\min} \dashrightarrow X_{\min}$.

The good thing is, on the other hand, that $f_{X_{\min}}$ satisfies the condition of *nearly étale map* to be defined below. So our result below is applicable to $f_{X_{\min}}$.

Theorem 75 (cf. [NZ, Theorem B])**.** *Let V be a normal projective variety with only canonical singularities such that $K_V \sim_{\mathbb{Q}} 0$. Let $h\colon V \dashrightarrow V$ be a dominant rational map which is nearly étale. Then there exist an abelian variety A, a weak Calabi–Yau variety F (see below), a finite étale morphism $\tau\colon F \times A \to V$, a nearly étale dominant rational map $\varphi_F\colon F \dashrightarrow F$, and a finite étale morphism $\varphi_A\colon A \to A$ such that $\tau \circ (\varphi_F \times \varphi_A) = h \circ \tau$, i.e., the*

diagram below is commutative:

$$\begin{array}{ccc} F \times A & \overset{\varphi_F \times \varphi_A}{\cdots\rightarrow} & F \times A \\ \tau \downarrow & & \downarrow \tau \\ V & \overset{h}{\cdots\rightarrow} & V. \end{array}$$

Remark 76.

(1) If F has only terminal singularities, then φ_F is étale in codimension one.
(2) If the algebraic fundamental group $\pi_1^{\mathrm{alg}}(F)$ is finite (this is true if $\dim F \le 3$ by [NS, Corollary (1.4)]), then φ_F is a birational automorphism. In particular, if V has only terminal singularities and $q^{\max}(V) = \dim V - 2$, then φ_F is an automorphism.

Definition 77. A normal projective variety F is called *weak Calabi–Yau* if F has only canonical singularities, $K_F \sim_{\mathbb{Q}} 0$, and

$$q^{\max}(F) := \max\{q(F') \mid F' \to F \text{ is finite étale}\} = 0.$$

If F is a nonsingular weak Calabi–Yau variety, then $\pi_1(F)$ is finite by Bogomolov's decomposition theorem, so a finite étale cover of F is expressed as a product of holomorphic symplectic manifolds and of Calabi–Yau manifolds. If F is singular, we have to use a result of Kawamata instead.

Definition 78. Let $h\colon V \cdots\rightarrow W$ be a proper rational (resp. meromorphic) map between algebraic (resp. complex analytic) varieties. The map h is called *nearly étale* if there exist proper birational (resp. bimeromorphic) maps $\mu\colon Y \cdots\rightarrow W$, $\nu\colon X \cdots\rightarrow V$ and a morphism $f\colon X \to Y$ such that

(1) X and Y are algebraic (resp. complex analytic) varieties,
(2) f is a finite étale morphism, and
(3) $\mu \circ f = h \circ \nu$.

As we have seen from the results above, the building blocks of surjective endomorphisms of nonsingular projective varieties with Kodaira dimension ≥ 0, are the endomorphisms of abelian varieties and nearly étale rational endomorphisms of weak Calabi-Yau varieties.

We still have to treat the case of Kodaira dimension $-\infty$. Let X be a nonsingular projective variety of Kodaira dimension $-\infty$ and $f : X \to X$ a surjective endomorphism. The study of such f is very difficult. For instance, we still do not know well the structure of such f even when X is a *rational* surface, though for surfaces X, the geometrical structure of the underlying surface X is well understood (provided that f is non-isomorphic), thanks to [Na] and [FN05].

On the other hand, it is conjectured that being $\kappa(X) = -\infty$ is equivalent to the uniruledness of X (*weak Abundance conjecture*). This conjecture is known if $\dim X \le 3$.

Therefore, the result below (which uses [BCHM]) and the weak Abundance conjecture reduce the study of $f : X \to X$ to that of an étale endomorphism $f_Y : Y \to Y$ of a non-uniruled projective manifold Y.

Indeed, the M in Theorem 79 below is birational to the Cartesian product $M' := M \times_Y Y$ of $\pi : M \to Y$ and $f_Y : Y \to Y$, and (X, f) and (M, f_M) are birationally equivariant to (M', pr_M) with $\mathrm{pr}_M : M' \to M$ the first projection (so f and f_M are birationally determined by f_Y); this is because rationally connected projective manifolds are simply connected.

For $h : Y \to Y$ we could apply Theorem 72 (assuming weak Abundance conjecture), and note that $\dim Y < \dim X$. This way, we have confirmed Claim 71.

Theorem 79 (cf. [NZ, Theorem C and its Remark])**.**
Let X be a projective manifold with an étale endomorphism f. Assume that X is uniruled. Lifting (X, f) birationally equivariantly to a pair (M, f_M) (with $f_M : M \to M$ étale too), we have a maximal rationally connected fibration $\pi : M \to Y$ onto a non-uniruled projective manifold Y and an étale endomorphism $f_Y : Y \to Y$ such that $\pi \circ f_M = f_Y \circ \pi$, i.e., the diagram below is commutative:

$$\begin{array}{ccc} M & \xrightarrow{f_M} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{f_Y} & Y. \end{array}$$

We refer to [Fu] and [FN07] for the precise geometrical structure of non-singular projective threefolds with non-negative Kodaira dimension and admitting non-isomorphic surjective endomorphisms.

We refer to [Zh2] or §5 for the building blocks of automorphisms of algebraic manifolds.

8. Cohomologically hyperbolic endomorphism

Let X be a compact Kähler manifold of dimension $n \geq 2$. A surjective endomorphism $f : X \to X$ is *cohomologically hyperbolic* in the sense of [Gu06], if there is an $\ell \in \{1, 2, \dots, n\}$ such that the ℓ-th dynamical degree

$$d_\ell(f) > d_i(f) \qquad \text{for} \quad \text{all} \quad (\ell \neq)\ i \ \in \ \{0, 1, \dots, n\},$$

or equivalently, for both $i = \ell \pm 1$, by the Khovanskii - Tessier inequality.

In his papers [Gu05] - [Gu06], Guedj assumed that a dominant rational self map $f : X \cdots\to X$ has *large topological degree* (i.e., it is cohomologically hyperbolic with $\ell = \dim X$ in the definition above), and constructed a *unique* f_*-invariant measure μ_f. Further, the measure is proved to be of maximal entropy, ergodic, equidistributive for f-periodic and repulsive points, and with strictly positive Lyapunov exponents. In [Gu06], Guedj classified cohomologically hyperbolic rational self maps of surfaces S and deduced that the Kodaira dimension $\kappa(S) \leq 0$. *Then he conjectured that the same should hold in higher dimension.*

The result below gives an affirmative answer to the above-mentioned conjecture of Guedj [Gu06] page 7 for holomorphic endomorphisms (see [Zh2, Theorem 1.3] for the case of automorphisms on threefolds). The proof is given by making use of results in [NZ] or Theorem 74 above. It is classification-free and for arbitrary dimension.

Theorem 81 (cf. [Zh3])**.** *Let X be a compact complex Kähler manifold and $f : X \to X$ a surjective and cohomologically hyperbolic endomorphism. Then the Kodaira dimension $\kappa(X) \leq 0$.*

We now determine the geometric structure for projective threefolds in the above theorem.

Theorem 82 (cf. [Zh3]). *Let V be a smooth projective threefold and let $f\colon V \to V$ be a surjective and cohomologically hyperbolic étale endomorphism. Then one of the following cases occurs.*

(1) *V is f-equivariantly birational to a* Q*-torus, i.e., the quotient of an abelian variety modulo a finite and free action.*
(2) *V is birational to a weak Calabi-Yau variety, and $f \in \mathrm{Aut}(V)$.*
(3) *V is rationally connected, and $f \in \mathrm{Aut}(V)$.*
(4) *The albanese map $V \to \mathrm{Alb}(V)$ is a smooth and surjective morphism onto the elliptic curve $\mathrm{Alb}(V)$ with every fibre a smooth projective rational surface of Picard number ≥ 11. Further, the dynamical degrees satisfy $d_2(f) > d_1(f) \ge \deg(f) \ge 2$.*
(5) *V is f-equivariantly birational to the quotient space of a product* (Elliptic curve)$\times$ $(K3)$ *modulo a finite and free action. Further, the dynamical degrees satisfy $d_2(f) > d_1(f) \ge \deg(f) \ge 2$.*

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